Tutorial Note XI

1 Gram-Schmidt Orthogonalization Procedure

In this section, we introduce the Gram-Schmidt orthogonalization procedure. You may have learned it in linear algebra, while we discuss it for functions here, which is the idea of functional analysis. In an inner product space, for vectors X_1, X_2, ..., which are assumed linearly independent, the Gram-Schmidt orthogonalization procedure gives us vectors Y_1, Y_2, ..., which are orthogonal and

$$\langle X_1, X_2, \dots, X_n \rangle = \langle Y_1, Y_2, \dots, Y_n \rangle,$$

where $\langle Z_1, Z_2, \dots, Z_n \rangle$ means the vector space spanned by Z_1, Z_2, ..., Z_n. The construction is as follows:

$$\begin{aligned}
Y_1 &= X_1, \\
Y_2 &= X_2 - \frac{(X_2, Y_1)}{(X_1, Y_1)} Y_1, \\
Y_3 &= X_3 - \frac{(X_3, Y_1)}{(X_1, Y_1)} Y_1 - \frac{(X_3, Y_2)}{(X_2, Y_2)} Y_2, \\
&\dots,
\end{aligned}$$

where $(\cdot, \cdot)$ denotes the inner product. In a Hilbert space H, given linearly independent $\{e_i\}$ satisfying that $\overline{\langle e_i \rangle} = H$, we could use the Gram-Schmidt orthogonalization procedure to obtain an orthonormal basis $\{f_i\}$. Next, as an exercise, we perform this procedure in $L^2[-1, 1] = \{f \mid \int_{-1}^{1} f^2 < \infty\}$ with inner product $(f, g) = \int_{-1}^{1} fg$ for 1, x, x^2, ... The new functions p_i are

$$\begin{aligned}
p_1 &= 1, \\
p_2 &= x - \frac{(x, 1)}{(1, 1)} 1 = x, \\
p_3 &= x^2 - \frac{(x^2, 1)}{(1, 1)} 1 - \frac{(x^2, x)}{(x, x)} x = x^2 - \frac{1}{3}, \\
p_4 &= x^3 - \frac{(x^3, 1)}{(1, 1)} 1 - \frac{(x^3, x)}{(x, x)} x - \frac{\left(x^3, x^2 - \frac{1}{3}\right)}{\left(x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\right)} \left(x^2 - \frac{1}{3}\right) = x^3 - \frac{3}{5} x,
\end{aligned}$$

....

If we normalize them by making $p_i(1) = 1$, they are called Legendre polynomials. By the Weierstrass approximation theorem, $\overline{\langle x^i \rangle} = L^2[-1,1]$, so Legendre polynomials is an orthogonal basis of $L^2[-1,1]$.

2 Sturm-Liouville Theory

In this section, a sketchy introduction of the Sturm-Liouville theory will be given. The Sturm-Liouville problem refers to, for example, the following ODE problem:

$$\begin{cases} -(pu)' + qu = \lambda u; \\ u(a) = u(b) = 0, \end{cases}$$

where $p(x) > 0$. The method to deal with this problem is similar to that to deal with PDEs. In fact, notice that the Sturm-Liouville problem is an elliptic problem in 1D. The method we present could also be used to deal with elliptic equations in high dimensions, see, for instance, Evans's PDE chapter 6. We use the following problem:

$$\begin{cases} -u'' + u = \lambda u; \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

as an example to illustrate the method. In fact, this problem is equivalent to

$$\begin{cases} -u'' = \lambda u; \\ u(a) = u(b) = 0. \end{cases}$$

The function of the term u will be seen later. First we establish a weak formulation of the problem

$$\begin{cases} -u'' + u = f; \\ u(a) = u(b) = 0. \end{cases} \tag{2}$$

By multiplying the equation by v and integration by parts, we obtain

$$\int u'v' + \int uv = \int fv \tag{3}$$

for all v satisfying $v(a) = v(b) = 0$. Conversely, the equation (3) also implies the original equation (2). So we have a weak formulation of the problem (2): finding u satisfying $u(a) = u(b)$ such that

$$\int u'v' + \int uv = \int fv$$

for all v satisfying $v(a) = v(b) = 0$. In fact, different boundary conditions have different formulations. For example, the Neumann problem:

$$\begin{cases} -u'' + u = f; \\ u'(a) = u'(b) = 0, \end{cases}$$

is equivalent to

$$\int u'v' + \int uv = \int fv$$

for all v. By, for example, the Lax-Milgram theorem in functional analysis, the existence of (3) is guaranteed. The term u is used to make this step available, especially for the problems such as the Neumann problem. We denote the solution operator $f \to u$ by $\mathscr{L}^{-1}$, then the problem (1) turns to be

$$\mathscr{L}^{-1}(\lambda u) = u.$$

So the problem (1) becomes an eigenvalue problem for $\mathscr{L}^{-1}$. This problem could be solved by the theory of compact self-adjoint operators. We just present the conclusion here. It says that if $\mathscr{L}^{-1}$ is self-adjoint and compact (which could be neglected here), then the eigenfunctions forms an orthonormal basis, which is the foundation of the method of separation of variables. Here $\mathscr{L}^{-1}$ is self-adjoint means that

$$\int \mathscr{L}^{-1} f \bar{g} = \int f \overline{\mathscr{L}^{-1} g},$$

corresponding to the symmetric boundary conditions in Strauss's PDE. If you are interested in this theory, you may refer to Brezis's functional analysis chapter 8.